

# Diagonalizability of Constraint Propagation Matrices

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In order to obtain stable and accurate general relativistic simulations, re-formulations of the Einstein equations are necessary. In a series of our works, we have proposed using eigenvalue analysis of constraint propagation equations for evaluating violation behavior of constraints. In this article, we classify asymptotical behaviors of constraint-violation into three types (asymptotically constrained, asymptotically bounded, and diverge), and give their necessary and sufficient conditions. We find that degeneracy of eigenvalues sometimes leads constraint evolution to diverge (even if its real-part is not positive), and conclude that it is quite useful to check the diagonalizability of constraint propagation matrices. The discussion is general and can be applied to any numerical treatments of constrained dynamics.

## I. INTRODUCTION

So-called numerical relativity (computational simulations in general relativity) is a promising research field having implications for ongoing astrophysical observations such as gravitational wave astronomy [1]. Many simulations of binary compact objects have revealed that mathematically equivalent sets of evolution equations show different numerical stability in the free-evolution scheme.

There are many approaches to re-formulate the Einstein equations for obtaining a longterm stable and accurate numerical evolution (e.g. see references in [2]). In a series of our works, we have proposed the construction of a system that has its constraint surface as an attractor. By applying eigenvalue analysis of constraint propagation equations, we showed that there *is* a constraint-violating mode in the standard Arnowitt-Deser-Misner (ADM) evolution system [3, 4] when it is applied to a single non-rotating black-hole space-time[6]. We also found that such a constraint-violating mode can be compensated for if we adjust the evolution equations with a particular modification using constraint terms like the one proposed by Detweiler [5].

Our predictions are borne out in simple numerical experiments using the Maxwell, Ashtekar, and ADM systems [6, 7, 8, 9]. There are also several numerical experiments to confirm our predictions are effective[10, 11]. However we have not yet obtained definite guidelines for specifying the above adjusting terms and their multipliers.

In this article, we show the essential steps in analyzing constraint amplification factors (defined in §II B). In §III, we show that degeneracy of eigenvalues sometimes leads constraint evolution to diverge. This observation

suggests the importance of checking the diagonalizability of characteristic matrices, and gives further insights for constructing an asymptotically constrained system.

## II. A GUIDELINE TO OBTAIN A ROBUST EVOLUTION SYSTEM

### A. Idea of Adjusted system

We begin by reviewing our proposal for an “adjusted system”.

Suppose we have a dynamical system of variables  $u^a(x^i, t)$ , which has evolution equations,

$$\partial_t u^a = f(u^a, \partial_i u^a, \dots), \quad (2.1)$$

and the (first class) constraints,

$$C^\alpha(u^a, \partial_i u^a, \dots) \approx 0. \quad (2.2)$$

Note that we do not require (2.1) to form a first-order hyperbolic form. We propose to investigate the evolution equation of  $C^\alpha$  (constraint propagation),

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \dots), \quad (2.3)$$

for evaluating violation features of constraints.

The character of constraint propagation, (2.3), will vary when we modify the original evolution equations. Suppose we modify (adjust) (2.1) using constraints

$$\partial_t u^a = f(u^a, \partial_i u^a, \dots) + F(C^\alpha, \partial_i C^\alpha, \dots), \quad (2.4)$$

then (2.3) will also be modified as

$$\partial_t C^\alpha = g(C^\alpha, \partial_i C^\alpha, \dots) + G(C^\alpha, \partial_i C^\alpha, \dots). \quad (2.5)$$

Therefore, finding a proper adjustment  $F(C^\alpha, \dots)$  is a quite important problem.

Hyperbolicity analysis may be a way to evaluate constraint propagation, (2.3) and (2.5) [12]. However, this

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requires (2.3) to be a first-order system which is easy to be broken. (See e.g. Detweiler-type adjustment [5] in the ADM formulation [6]). Furthermore hyperbolicity analysis only concerns the principal part of the equation, that may fail to analyze the detail evaluation of evolution.

Alternatively, we have proceeded an eigenvalue analysis of the whole RHS in (2.3) and (2.5) after a suitable homogenization, which may compensate for the above difficulties of hyperbolicity analysis.

### B. CP matrix and CAF

We propose to transform the constraint propagation equation, (2.3) and (2.5), into Fourier modes,

$$\partial_t \hat{C}^\alpha = \hat{g}(\hat{C}^\alpha) = M^\alpha_\beta \hat{C}^\beta, \\ \text{where } C(x, t)^\alpha = \int \hat{C}(k, t)^\alpha \exp(ik \cdot x) d^3k, \quad (2.6)$$

then to analyze the eigenvalues, say  $\Lambda_\alpha$ , of the coefficient matrix,  $M^\alpha_\beta$ . We call  $\Lambda_\alpha$  and  $M^\alpha_\beta$  the constraint amplification factors (CAFs) and constraint propagation matrix (CP matrix), respectively.

So far we have proposed the following heuristic conjectures [6, 7, 8, 9]:

- (A) If the CAF has a *negative real-part* (the constraints are necessarily diminished), then we see more stable evolution than a system which has a positive CAF.
- (B) If the CAF has a *non-zero imaginary-part* (the constraints are propagating away), then we see more stable evolution than a system which has a zero CAF.

We observe that this eigenvalue analysis requires the fixing of a particular background space-time, since the CAFs depend on the dynamical variables,  $u^a$ .

### C. Classification of Constraint propagations

The CAFs indicate the evolution of constraint violations (definitely its Fourier modes). It is natural to assume that a divergence of constraint norm is related to the numerical blow-ups. Therefore we classify the fundamental evolution property of constraint propagation equation (2.6) as follows:

- (C1) *Asymptotically constrained* : Violation of constraints decays (converges to zero).
- (C2) *Asymptotically bounded* : Violation of constraints is bounded at a certain value.
- (C3) *Diverge* : At least one constraint will diverge.

Note that (C1)  $\subset$  (C2). We will derive the necessary and sufficient conditions for (C1) and (C2) in the next section.

## III. CONDITIONS FOR (C1) AND (C2)

### A. Preparation

Hereafter, we consider a set of evolution equations,

$$\partial_t C^i(t) = M^i_j C^j, \quad (3.1)$$

where  $C^i$  ( $i = 1, \dots, n$ ) is a complex-valued vector,  $M^i_j$  is a  $n \times n$  complex-valued matrix, and  $C^i(t)$  is assumed to have finite-valued initial data  $C^i(0)$ .

Without loss of generality, the CP matrix  $M$  can be assumed to be a Jordan normal form, since within complex-valued operations all the matrices can be converted to this form. Suppose that  $M$  has  $r$  different eigenvalues ( $\lambda_1, \dots, \lambda_r$ ), where  $r \leq n$ . Let the multiplicity of  $\lambda_k$  as  $n_k$ , where  $\sum_{k=1}^r n_k = n$ .  $M$  can be expressed as

$$M = J_1 \dot{+} \dots \dot{+} J_r := \begin{pmatrix} J_1 & & O \\ & \ddots & \\ O & & J_r \end{pmatrix}, \quad (3.2)$$

where the cell size of  $J_k$  is  $n_k \times n_k$ . The Jordan matrix  $J_k$  is then expressed using a Jordan block  $J_{k\ell}$ ,

$$J_k = J_{k1} \dot{+} \dots \dot{+} J_{km}, \quad (3.3)$$

$$J_{k\ell} := \begin{pmatrix} \lambda_k & 1 & & O \\ & \ddots & \ddots & \\ O & & \lambda_k & 1 \end{pmatrix}. \quad (3.4)$$

Note that  $J_{k\ell}$  is  $n_{k\ell} \times n_{k\ell}$ ,  $\sum_{\ell=1}^m n_{k\ell} = n_k$ ,  $m = n - \text{rank}(M - \lambda_k E)$  and  $\max_{\ell} (n_{k\ell}) = \nu_k$ . The minimum polynomial of  $M$  is written as

$$\mu_M(t) = (t - \lambda_1)^{\nu_1} \dots (t - \lambda_r)^{\nu_r}. \quad (3.5)$$

If  $J_k$  is diagonal (i.e.  $n_k = n - \text{rank}(M - \lambda_k E)$ ), then  $\nu_k = 1$  for that  $k$ . If  $M$  is diagonalizable (i.e.  $n_k = n - \text{rank}(M - \lambda_k E)$  for  $\forall k$ ), then  $\nu_k = 1$  for all  $k$ .

We then have the following statement.

**Proposition 1** *The solution of*

$$\partial_t C_a = J_k C_a \quad (3.6)$$

*can be expressed formally as*

$$C_a(t) = \exp(\lambda_k t) \sum_{\ell=0}^{\nu_k-1} (a_\ell^{(k)} t^\ell). \quad (3.7)$$

A proof is available by mathematical induction. Suppose that  $J_{k1}$  is  $\nu_k \times \nu_k$  which is the maximal size  $J_k$ . By direct calculation, we have that  $\partial_t C_a = J_{k1} C_a$  yields (3.7) with  $t$ -polynomial of degree  $(\nu_k - 1)$ . Then we see that (3.7) is satisfied in general.

From this proposition, the highest power  $N_k$  in  $t$ -polynomial in (3.7) is bounded by  $0 \leq N_k \leq \nu_k - 1$ . The matrix  $J_k$  in (3.6) can be directly extended to the full CP matrix,  $M$ , in (3.1). Therefore the highest power  $N$  in all constraints is bounded by

$$0 \leq N \leq \max_{1 \leq k \leq r} (n_k) - 1. \quad (3.8)$$

### B. Asymptotically Constrained CP

The following Propositions 2 and 3 give us the next theorem.

**Theorem 1** *Asymptotically constrained evolution (violation of constraints converges to zero) is obtained if and only if all the real parts of the CAFs are negative.*

**Proposition 2** *All the real part of CAFs are negative  $\Rightarrow$  Asymptotically constrained evolution.*

proof) We use the expression (3.7). If  $\Re(\lambda_k) < 0$  for  $\forall k$ , then  $C_i$  will converge to zero at  $t \rightarrow \infty$  no matter what the  $t$ -polynomial terms are.  $\square$

**Proposition 3** *Asymptotically constrained evolution  $\Rightarrow$  All the real parts of the CAFs are negative.*

proof) We show the contrapositive. Suppose there exists an eigenvalue  $\lambda_1$  of which the real-part is non-negative. Then we get  $\partial_t C_1 = \lambda_1 C_1$  of which the solution is  $C_1 = C_1(0) \exp(\lambda_1 t)$ .  $C_1$  does not converge to zero.  $\square$

### C. Asymptotically Bounded CP

The following Propositions 4 and 5 give us the next theorem.

**Theorem 2** *Asymptotically bounded evolution (all the constraints are bounded at a certain value) is obtained if and only if all the real parts of CAFs are not positive and  $J_k$  is diagonal when  $\Re(\lambda_k) = 0$ .*

**Corollary** *Asymptotically bounded evolution is obtained if the real parts of CAFs are not positive and the CP matrix  $M^\alpha_\beta$  is diagonalizable.*

**Proposition 4** *All the real parts of CAFs are not positive and  $J_k$  is diagonal when  $\Re(\lambda_k) = 0 \Rightarrow$  Asymptotically bounded evolution.*

proof) We use the expression (3.7). When  $\Re(\lambda_k) < 0$ ,  $\exp(\lambda_k t) \times (t\text{-polynomials})$  will converge to zero no matter what the  $t$ -polynomial terms are. When  $\Re(\lambda_k) = 0$ , we see  $\nu_k = 1$  from the assumption of diagonality of  $J_k$ . So we see the  $t$ -polynomial terms are constant and  $\exp(\lambda_k t)$  is bounded.  $\square$

**Proposition 5** *Asymptotically bounded evolution  $\Rightarrow$  All the real parts of the CAFs are not positive and  $J_k$  is diagonal when  $\Re(\lambda_k) = 0$ .*

proof) We show the contrapositive. If there exists an eigenvalue of which the real-part is positive, then constraints will diverge no matter what the  $t$ -polynomial terms are. Therefore we try to show that constraints will diverge when all the real-parts of eigenvalues are non-positive, and there exists  $\lambda_k$  such that  $\Re(\lambda_k) = 0$  and its Jordan matrix  $J_k$  is not diagonal.

Since Jordan matrix  $J_k$  is not diagonal, we see the power of  $t$ -polynomial  $\nu_k$  is greater than 1 in the expression (3.7). Thus we have that (3.7) will diverge in  $t \rightarrow \infty$ .  $\square$

## IV. CONCLUDING REMARKS

Two theorems will give us a guideline to analyze a constraint-violating mode of the system. The result supports our previous heuristic conjecture (A), but also suggests an ill-behaving case when CAFs are degenerated and its real-part is zero, when the associated Jordan matrix is not diagonal. This indicates the importance of checking the diagonalizability of constraint propagation matrix  $M$ .

Along the line of our evaluation of constraint propagation equations (3.1), we propose a practical procedure for this classification in Figure 1. We think that this diagram will provide systematic predictions for obtaining a robust evolution system in any constrained dynamics.

The present classification is only on the fixed background spacetime and only for  $t \rightarrow \infty$ . It is still not clear at what value the constraints are bounded if a limiting value exists. Thus further modifications are underway. We are also applying the present classification scheme to various adjusted systems of the Einstein equations (adjusted ADM, and further modified versions), together with numerical experiments. We hope to report on them in the near future.

The current constraint analysis only concentrates to the evolution equations and does not include the effect of the boundary treatments. Since the eigenvalues are evaluated locally, it will be possible to include the effect of numerical boundary conditions if they are expressed ap-

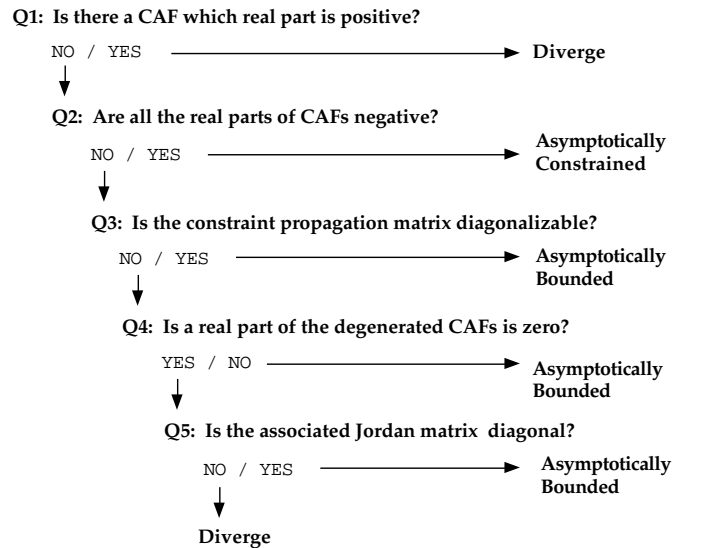


FIG. 1: A flowchart to classify the fate of constraint propagation.

parently in a part of the evolution equations. This is also the one direction to proceed our future research. Meanwhile, we would like to remark that one of our proposed adjustments in [9] contributes to enforce the computational ability of the black-hole excision boundary treatment [11].

By extending the notion of “norm” or “compactness” of constraint violations, it might be interesting to define a new measure which monitors a “distance” between the constraint surface and an evolution sector in constraint dynamics.

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